The Change in the Prefactor Exponent with Valence for Percolation in 3 and 4 Dimensions

J. A. M. S. Duarte

Solid State Group-Physics Department, Imperial College, Prince Consort Road, London SW 7 U.K.

and H. J. Ruskin

Metabolic Unit, Hunts House, Guy's Hospital, Medical School, London SE 1, U.K.

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The change of the prefactor exponent for restricted valence percolation in 3 and 4 dimensions is shown to occur between valences 2 and 3 as in 2 dimensions. For valence 2 the exponent is SAW-like and after the change it assumes the normal percolative value for each dimension.

One of the specializations of percolation in recent times has centered upon the internal structure of the clusters.

Hence, several studies of the cyclomatic number of clusters have used the number of neighbours to a site which belong to the cluster (or site valence) as the natural basis for calculation (Cherry [1], Cherry and Domb [2]).

A systematic study of the cluster statistics along the traditional lines, however, only started with the publication of Gaunt et al. [3].

The number of animals with restricted valence for site percolation was there found to undergo a change between valences 2 and 3 for the simple quadratic lattice.

Later the same authors studied the site problem on the triangular lattice with identical conclusions [4].

Now, valence in itself is as much a natural candidate to quantify the compactness of a cluster as any of the recently proposed alternatives. In this context, we have derived perimeter polynomials with restricted valence for the simple cubic, body-centered cubic and four-dimensional hypercubic lattice through order one less than that presently available for normal percolation (Sykes et al. [5], Gaunt et al. [6]) so that changes in the histogram structure can be followed (Duarte [7]). The restricted condition in each case is neatly achieved by providing, in the usual counting methods (Martin [8]), for the rejection of clusters incorporating space-types corresponding to coordination numbers higher than the required valence.

A valuable by-product is the total number of animals with restricted valence (V) given in Table 1 for the three lattices.

Gaunt et al. [3] gave an argument in favour of a change of the prefactor exponent θ for the total number (A_n) of site animals with n sites

$$A_n \sim \lambda^n n^{-\theta}, \qquad (1)$$
$$\lambda = \exp[\lim n^{-1} \ln A_n],$$

occurring somewhere between valences 2 and $d+1$ (d the lattice dimensionality). θ would equal the value for the asymptotically dominating neighbour-avoiding walks for valence 2 and would be equal to its normal percolation value for valences greater than $d+1$.

This implies the transition observed between valences 2 and 3 in two dimensions and opens up other possibilities in higher dimensionalities.

The present data can be checked, at least partially, with a number of previously derived results. They agree with Hioe [9] for the simple cubic neighbour avoiding walks and with Torrie and Whittington [10] for the body-centered cubic walks, once the contributions of strongly embedded polygons are accounted for. And, of course, for the total number of animals in normal percolation, further data are available, as mentioned before (Sykes et al. [5], Gaunt et al. [6]).

We have made a standard analysis using ratio and Padé techniques assuming a form of the type (1).

Reprint requests to J. A. M. S. Duarte, Laboratório de Física, Faculdade de Ciências, Universidade do Porto, 4000 Porto, Portugal.

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Table 1. Total number of restricted valence site animals for the s.c., b.c.c. and 4 D hypercubic lattices.

Sites n	$V = 2$	$V = 3$	$V = 4$	$V = 5$	$V = 6$	$V = 7$	$V = 8$
			Simple cubic lattice				
1	1	1	1	1	1		
2	3	3	3	3	3		
3	15	15	15	15	15		
4	66	86	86	86	86		
5	267	519	534	534	534		
6	1111	3247	3475	3481	3481		
7	4623	20807	23399	23501	23502		
8	19098	136393	161641	162895	162913		
9	78987	911439	11394221	1152639	1152870		
10	324789	6189701	8163899	8292218	8294738		
			Body centred cubic lattice				
1	1	1	1	1	1	1	1
2	4	4	4	4	4	4	4
3	28	28	28	28	28	28	28
4	160	216	216	216	216	216	216
5	820	1720	1790	1790	1790	1790	1790
6	4496	14180	15531	15531	15531	15531	15531
7	24628	119740	139502	140718	140746	140746	140746
8	133516	1036780	1285788	1305252	1305912	1305920	1305920
9	724036	9156740	12093424	12362624	12373868	12374068	12374069
			4 D-Hypercubic lattice				
1	1	1	1	1	1	1	1
2	4	4	4	4	4	4	4
3	28	28	28	28	28	28	28
4	178	234	234	234	234	234	234
5	1060	2092	2162	2162	2162	2162	2162
6	6452	19616	21216	21272	21272	21272	21272
7	39196	189856	217232	218712	218740	218740	218740
8	236746	1887818	2294922	2322906	2323722	2323730	2323730

Table 2. Estimates for λ and θ for restricted valence percolation in three and four dimensions.

λ	$V = 2$	$V = 3$	$V = 4$	$V = 5$	$V = 6$	$V = 7$	$V = 8$
Simple cubic	4.045 ± 0.008	7.86 ± 0.05	8.31 ± 0.05	8.33 ± 0.06	8.35 [5] ± 0.04		
Body-centered cubic	5.325 ± 0.015	10.45 ± 0.19	11.1 ± 0.1	11.16 ± 0.08	11.18 ± 0.08	11.18 ± 0.08	11.19 [5] ± 0.06
Hypercubic	~ 5.5	12.5 ± 0.5	13.2 ± 0.3	13.3 ± 0.25	13.3 ± 0.25	13.3 ± 0.25	13.35 ± 0.2 [6]
θ							
Simple cubic	$-\frac{1}{6} \pm \frac{1}{6}$	1.5 ± 0.1	1.5 ± 0.08	1.5 ± 0.08	1.55 [6] ± 0.05		
Body-centered cubic	$-\frac{1}{6} \pm \frac{1}{6}$	1.5 ± 0.15	1.5 ± 0.1	1.5 ± 0.08	1.5 ± 0.08	1.5 ± 0.08	$\sim 3/2$ [5]
Hypercubic	—	1.7 ± 0.4	1.9 ± 0.25	1.9 ± 0.25	1.9 ± 0.25	1.9 ± 0.25	1.9 [6] ± 0.15

Our estimates for λ and θ are given in Table 2. On these results we shall add a number of remarks.

a) For the b.c.c. and the s.c. lattices the argument of Watson [11] in favour of a value for $\theta \sim -\frac{1}{6}$ (and identical with that for the self-avoiding walk) seems consistent with the data presented. Padé results show a dispersion of residues typically around -0.05 to -0.25. If the conjectured value is used to determine biased estimates for λ, they would seem to favour the lower half of the estimate given by Torrie and Whittington [10]. Hence our central value is slightly lower than theirs for the b.c.c. lattice.

b) The transition between self-avoiding walk and percolative behaviour is clearly shown for both 3-dimensional lattices to occur between valences 2 and 3. Thus, we have used the percolative estimate for θ to improve the estimates for λ.

c) For valence 2 on the 4-dimensional hypercubic lattice we carried out standard analyses only. It is natural to expect for these animals to behave asymptotically as the self-avoiding walks, which possess a confluent logarithmic term (McKenzie [12]). Standard techniques are known to lead to slightly erroneous estimates and spurious exponents (Guttmann [13]), whenever enough terms are available. Hence, we have decided to push the expansion further, before refining the analysis with Guttmann's method.

d) For valences $\geqq 3$, the prefactor exponent is of the same nature as that for normal percolation. It is likely that the asymptotic form (1) is too simple and that confluent singularities will impair the rate of convergence as for normal percolation (Gaunt et al. [6]). Nevertheless, ratio method estimates would seem to ensure $\theta > 1.55$ (the value for 3-dimensions) for all valences $\geqq 4$.

Convergence gets poorer for the lower valences, as is especially the case for valence 3, and our estimate ranges are considerably wider than those given for the other two lattices.

In summary, we think it can be assumed on the present evidence that the change of the pre-factor exponent occurs for restricted valence percolation in 3 and 4 dimensions between valences 2 and 3, and that the data are broadly in favour of it assuming the percolative value for all valences $\geqq 3$. These conclusions are completely in line with those found for the 2-dimensional case*.

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* *Note added in proof:* After refereeing, we became aware of the extension of Refs. (3) and (4) to the simple cubic lattice (D.S. Gaunt, J. Martin, E. Ord, S.G. Wittington, G.M. Torrie (1980) in the press) in complete agreement with the present results.

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